Multilevel Upscaling of Heterogeneous Media

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The modeling of flow in porous media touches many important aspects of our everyday lives, from sustaining and protecting our subsurface aquifers to enhancing the production of petroleum reservoirs. Effective simulation of these multiscale and multicomponent flows, however, is inhibited by fundamental mathematical and algorithmic challenges. One such challenge is the need to resolve the multiscale structure of geological formations; the length scales observed in sedimentary laminae range from the millimeter scale upward, while the simulation domain may be on the order of several kilometers. Fully resolved simulations are, thus, computationally intractable, yet the fine-scale variations of the model parameters (e.g., structure and orientation of laminae) significantly affect the properties of the solution at all scales. This complex interaction of different length scales is not unique to flows in porous media, but arises in many other disciplines, including composite material design and analysis, hurricane and wild-fire modeling, and atmospheric and ocean circulation models.

The objective of a classical upscaling or homogenization procedure is to define an approximate mathematical model in which the *effective* properties of the medium vary on a scale suitable for efficient computation. To do this, the macroscopic flow model, with parameters that vary on the microscopic (or fine) scale, is *averaged*, in some sense, over the microscopic length scales (see [1] and references therein). This approach has proved useful for modeling single-phase flow in mildly heterogeneous porous media; however, both strongly heterogeneous media and multiphase flow remain problematic.

In this research, we explore a new multilevel upscaling (MLUPS) methodology that accurately and efficiently treats the multiscale properties of the underlying porous medium and flow model. MLUPS is based on a generalization of the multigrid homogenization (MGH) algorithm developed in [1]. The MGH approach builds on the observation that the operator-dependent variational coarsening central to robust multigrid algorithms can also be viewed as an upscaling procedure.

In the MGH procedure, however, the focus is on the coarsest scale and the fact that the variational coarsening procedure generates a complete and self-consistent hierarchy of coarse-scale models, with their corresponding basis functions, is neglected. In the MLUPS method, this hierarchy is created by taking the fine-scale discretization and using BoxMG (see [1] and references therein) to coarsen it to a specified computational scale. This operator-induced variational coarsening effectively reduces the dimension of the fine-scale operator by selecting an appropriate local, low-energy basis for the coarse-scale. The coarse-scale model is solved, with this solution yielding a fine-scale representation via the multiscale basis functions. This approach provides a natural setting for

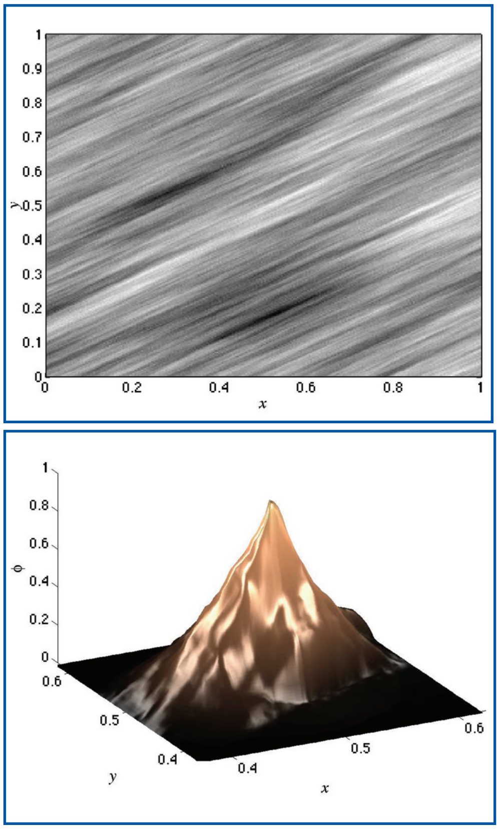

***Figure 1—* A realization of a strongly heterogeneous permeability field with variation (from light to dark) of 6 orders of magnitude (top). Our multilevel upscaling algorithm constructs a self-consistent hierarchy of coarse-scale models for single-phase saturated flow, as well as the corresponding multiscale basis functions, without solving any local or global fine-scale problems. The multiscale basis function for the center of the domain, shown in the lower figure, was generated using this algorithm. The fine-scale structure is clearly visible in the surface, which accurately represents the influence of this structure on the flow.**

adaptivity, error estimation, and extensions to more complex regimes such as unsaturated, multiphase, and reactive flows.

We consider a permeability field generated by the GSLIB software package [2]. This field, shown in the first figure, has a range of permeabilities from approximately 10^{-3} (light) to 10^{3} (dark). A coarse-scale pressure gradient is imposed on a fine computational grid of 256 × 256 elements, with impermeable boundary conditions on the top and bottom edges, to induce flow from left to right. We compare the results of the MLUPS method with the current state of the art, the Multiscale Finite Element Method (MSFEM) [3], for a coarse computational scale of 8 × 8 elements.

Errors in both the average flux across the line $x = x_1$, $\mathbf{q}(x_1)$, and the pressure, $p(x,y)$, are shown in Table 1. A 2048 × 2048 grid calculation, which predicts a constant flux in the x-direction of 1.13, is used to represent the true solution of the PDE, while an important benchmark is the bilinear finite element (BLFEM) solution on the 256 × 256 grid that indicates the "best" accuracy that we can, in general, expect at the fine computational scale. This computation takes 1.94 s on a 1.6 Ghz Athlon machine, only slightly more than the 1.88 s required by MSFEM, while the MLUPS computation requires only 0.18s, less than one tenth of the MSFEM cost.

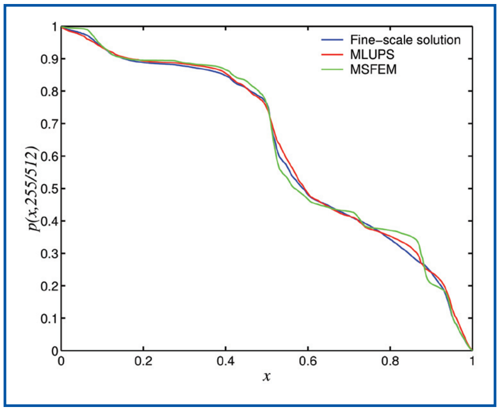

Figure 2— ***Both methods follow general trends in the pressure quite well. MSFEM, however, exhibits more significant localized deviations from the true pressure, induced by the artificial boundary conditions used to determine the basis functions.***

[1] J.D. Moulton, J.E. Dendy, and J.M. Hyman, "The Black Box Multigrid Numerical Homogenization Algorithm," *J. Comput. Phys.* **141**,1–29 (1998).

[2] C. Deutsch and *A. Journel, GSLIB, Geostatistical Software Library and User's Guide*, Second Edition (Oxford University Press, Oxford, 1998).

[3] T. Hou and X. Wu, "A Multiscale Finite Element Method for Elliptic Problems in Composite Materials and Porous Media," *J. Comput. Phys.* **134**, 169–189 (1997).

Table 1— ***Errors in flow properties.***

Measure	BLFEM	MSFEM	MLUPS
$\left\|e(\mathbf{q})\right\|_\infty$	2.96×10^{-2}	3.32×10^{-1}	2.08×10^{-1}
$\left\|e(\mathbf{q})\right\|_2$	2.96×10^{-2}	1.55×10^{-1}	1.06×10^{-1}
$\left\|e(p)\right\|_\infty$	1.20×10^{-2}	8.38×10^{-2}	9.52×10^{-2}
$\left\|e(p)\right\|_2$	8.92×10^{-4}	1.33×10^{-2}	9.58×10^{-3}